WEAK TYPE ESTIMATES OF THE MAXIMAL QUASIRADIAL BOCHNER-RIESZ OPERATOR ON CERTAIN HARDY SPACES

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Abstract. Let $\{A_t\}_{t>0}$ be the dilation group in $\mathbb{R}^n$ generated by the infinitesimal generator M where $A_t = \exp(M \log t)$, and let $\varrho \in C^\infty(\mathbb{R}^n \setminus \{0\})$ be a A_t-homogeneous distance function defined on $\mathbb{R}^n$. For $f \in \mathfrak{S}(\mathbb{R}^n)$, we define the maximal quasiradial Bochner-Riesz operator $\mathfrak{M}^\delta_\varrho$ of index $\delta > 0$ by

$$\mathfrak{M}^\delta_\varrho f(x) = \sup_{t>0} \left| \mathcal{F}^{-1}[(1-\varrho/t)^\delta_+ \hat{f}\,](x) \right|.$$

If $A_t = t\,I$ and $\{\xi \in \mathbb{R}^n | \, \varrho(\xi) = 1\}$ is a smooth convex hypersurface of finite type, then we prove in an extremely easy way that $\mathfrak{M}^\delta_\varrho$ is well defined on $H^p(\mathbb{R}^n)$ when $\delta = n(1/p - 1/2) - 1/2$ and $0 < p < 1$; moreover, it is a bounded operator from $H^p(\mathbb{R}^n)$ into $L^{p,\infty}(\mathbb{R}^n)$.

If $A_t = t\,I$ and $\varrho \in C^\infty(\mathbb{R}^n \setminus \{0\})$, we also prove that $\mathfrak{M}^\delta_\varrho$ is a bounded operator from $H^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ when $\delta > n(1/p - 1/2) - 1/2$ and $0 < p < 1$.

1. Introduction.

Let $\mathfrak{S}(\mathbb{R}^n)$ be the Schwartz space on $\mathbb{R}^n$. For $f \in \mathfrak{S}(\mathbb{R}^n)$, we denote the Fourier transform of f by

$$\mathcal{F}[f](x) = \hat{f}(x) = \int_{\mathbb{R}^n} e^{-i\langle x, \xi\rangle} f(\xi)\, d\xi.$$

Then the inverse Fourier transform of f is given by

$$\mathcal{F}^{-1}[f](x) = \check{f}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x, \xi\rangle} f(\xi)\, d\xi.$$

Let M be a real-valued $n \times n$ matrix whose eigenvalues have positive real parts. Then we consider the dilation group $\{A_t\}_{t>0}$ in $\mathbb{R}^n$ generated by the infinitesimal generator M, where $A_t = \exp(M \log t)$ for $t > 0$. We introduce A_t-homogeneous distance functions ϱ defined on $\mathbb{R}^n$; that is, $\varrho : \mathbb{R}^n \to [0, \infty)$ is a continuous function satisfying $\varrho(A_t \xi) = t\varrho(\xi)$ for all $\xi \in \mathbb{R}^n$. One can refer to [3] and [11] for its fundamental properties.

In what follows we shall denote by $\Sigma_\varrho \doteqdot \{\xi \in \mathbb{R}^n | \, \varrho(\xi) = 1\}$ the unit sphere of ϱ and denote by $\mathbb{R}^n_0 = \mathbb{R}^n \setminus \{0\}$. We use the polar coordinates; given $x \in \mathbb{R}^n$, we write $x = r\theta$ where $r = |x|$ and $\theta = (\theta_1, \theta_2, \cdots, \theta_n) \in S^{n-1}$. Given two quantities A and B, we write $A \lesssim B$ or $B \gtrsim A$ if there is a positive constant c (possibly depending on the dimension n and the index p to be given) such that $A \leq cB$. We also write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$.

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For $f \in \mathfrak{S}(\mathbb{R}^n)$, we consider quasiradial Bochner-Riesz means of index $\delta > 0$ defined by

$$\mathfrak{R}^{\delta}_{\varrho,t} f(x) = \mathcal{F}^{-1}[(1-\varrho/t)^{\delta}_{+} \hat{f}\,](x),$$

and the corresponding maximal operator

$$\mathfrak{M}^{\delta}_{\varrho} f(x) = \sup_{t>0} \left| \mathfrak{R}^{\delta}_{\varrho,t} f(x) \right|.$$

In the special case that $\varrho(\xi) = |\xi|^2$ and $A_t = t\,I$, Stein, Taibleson, and Weiss [10] proved that if $0 < p < 1$, then $\mathfrak{M}^{\delta}_{\varrho}$ is bounded from $H^p(\mathbb{R}^n)$ into $L^{p,\infty}(\mathbb{R}^n)$ at the critical index $\delta = \delta(p) \fallingdotseq n(1/p - 1/2) - 1/2$ where $H^p(\mathbb{R}^n)$ is the standard real Hardy space defined in Stein [9] and $L^{p,\infty}(\mathbb{R}^n)$ is one of the Lorentz spaces (which is called weak-L^p space) defined in Stein and Weiss [12] and furthemore Stein obtained the exceptional result that there is $f \in H^1(\mathbb{R}^n)$ such that a.e. convergence of the Bochner-Riesz means fails for $p = 1$ and $\delta(1) = (n-1)/2$.

In our first result we shall assume that $\varrho \in C^{\infty}(\mathbb{R}^n_0)$, $A_t = t\,I$ and Σ_ϱ is a smooth convex hypersurface of $\mathbb{R}^n$ which is of finite type, i.e. every tangent line makes finite order of contact with Σ_ϱ. We say that Σ_ϱ is of finite type $k \geq 2$ if k is the maximal order of contact on Σ_ϱ.

Theorem 1.1. *Suppose that $A_t = t\,I$, $\varrho \in C^{\infty}(\mathbb{R}^n_0)$ is a A_t-homogeneous distance function defined on $\mathbb{R}^n$, and Σ_ϱ is a smooth convex hypersurface of finite type. Then $\mathfrak{M}^{\delta(p)}_{\varrho}$ is well defined on $H^p(\mathbb{R}^n)$ when $0 < p < 1$; moreove, $\mathfrak{M}^{\delta(p)}_{\varrho}$ is a bounded operator from $H^p(\mathbb{R}^n)$ into $L^{p,\infty}(\mathbb{R}^n)$. That is, there is a constant $C = C(n,p,\Sigma_\varrho) > 0$ such that for any $f \in H^p(\mathbb{R}^n)$,*

$$\left| \{ x \in \mathbb{R}^n \,|\, \mathfrak{M}^{\delta(p)}_{\varrho} f(x) > \lambda \} \right| \leq \frac{C}{\lambda^p} \, \|f\|^p_{H^p(\mathbb{R}^n)}, \ \lambda > 0,$$

where $|E|$ denotes the Lebesgue measure of the set $E \subset \mathbb{R}^n$.

Remark. As a matter of fact, we prove this result under more general surface condition than the finite type condition on Σ_ϱ, which is to be called a spherically integrable condition of order < 1 in Section 3.

Our second result is to obtain that if $\delta > n(1/p - 1/2) - 1/2$ and $0 < p < 1$ then $\mathfrak{M}^{\delta}_{\varrho}$ admits (H^p, L^p)-estimate under no surface condition on Σ_ϱ.

Theorem 1.2. *Suppose that $A_t = t\,I$ and $\varrho \in C^{\infty}(\mathbb{R}^n_0)$ is a A_t-homogeneous distance function defined on $\mathbb{R}^n$. If $\delta > \delta(p)$ for $0 < p < 1$, then $\mathfrak{M}^{\delta}_{\varrho}$ is a bounded operator from $H^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$; that is, there is a constant $C = C(n,p) > 0$ such that for any $f \in H^p(\mathbb{R}^n)$,*

$$\|\mathfrak{M}^{\delta}_{\varrho} f\|_{L^p(\mathbb{R}^n)} \leq C\, \|f\|_{H^p(\mathbb{R}^n)},$$

provided that $\delta > n(1/p - 1/2) - 1/2$ and $0 < p < 1$.

Remark. This problem is still left open on the critical index $\delta = n(1/p - 1/2) - 1/2$ and $0 < p < 1$.

2. (H^p, L^p)-estimate for the case that $\varrho \in C^{\infty}(\mathbb{R}^n_0)$ and $\delta > \delta(p)$.

We shall employ a decomposition of the Bochner-Riesz multiplier $(1-\varrho)^{\delta}_{+}$ as in A. Córdoba [2]. Let $\phi \in C^{\infty}_0(1/2, 2)$ satisfy $\sum_{k\in\mathbb{Z}} \phi(2^k t) = 1$ for all $t > 0$. For $k \in \mathbb{N}$, let $\Phi^{\delta}_k = \phi(2^{k+1}(1-\varrho))(1-\varrho)^{\delta}_{+}$ and $\Phi^{\delta}_0 = (1-\varrho)^{\delta}_{+} - \sum_{k\in\mathbb{N}} \Phi^{\delta}_k$. For each $k \in \mathbb{Z}$, we

now introduce a partition of unity $\Xi_{k\ell}, \ell = 1, 2, \cdots, N_k$, on the unit sphere Σ_ϱ which we extend to $\mathbb{R}^n$ by way of $\Pi_{k\ell}(A_t\zeta) = \Xi_{k\ell}(\zeta), t > 0, \zeta \in \Sigma_\varrho$, and which satisfies the following properties; there are a finite number of points $\zeta_{k1}, \zeta_{k2}, \cdots, \zeta_{kN_k} \in \Sigma_\varrho$ such that for $\ell = 1, 2, \cdots, N_k$,

(i) $\sum_{\ell=1}^{N_k} \Pi_{k\ell}(\zeta) \equiv 1$ for all $\zeta \in \Sigma_\varrho$,

(ii) $\Xi_{k\ell}(\zeta) = 1$ for all $\zeta \in \Sigma_\varrho \cap B(\zeta_{k\ell}; 2^{-k/2})$,

(iii) $\Xi_{k\ell}$ is supported in $\Sigma_\varrho \cap B(\zeta_{k\ell}; c_1\, 2^{-k/2})$,

(iv) $|\mathcal{D}^\alpha \Pi_{k\ell}(\xi)| \le c_2\, 2^{|\alpha|k/2}$ for any multiindex α, if $1/2 \le \varrho(\xi) \le 2$,

(v) $N_k \le c_3\, 2^{(n-1)k/2}$ for fixed k,

where $B(\zeta_0; s)$ denotes the ball in $\mathbb{R}^n$ with center $\zeta_0 \in \Sigma_\varrho$ and radius $s > 0$ and the positive constants c_1, c_2, c_3 do not depend upon k. For each $k \in \mathbb{Z}$, let $\mathcal{H}^\delta_{\varrho k\ell} = \mathcal{F}^{-1}[\Phi^\delta_k \Pi_{k\ell}]$ and $\mathcal{H}_0 = \mathcal{F}^{-1}[\Phi^\delta_0]$.

Next we invoke a simple observation used in [8] to obtain decay estimate for kernels $\mathcal{H}_{k\ell}, \mathcal{H}_0$ corresponding to the decomposition of the Bochner-Riesz multiplier defined in the above. Without loss of generality, we can assume that $\varrho \in C^\infty(\mathbb{R}^n)$ because we can replace ϱ by ϱ^N for sufficiently large $N > 0$ by a subordination argument in [3]. Then we easily see that the kernel $\mathcal{H}_0$ has a nice decay, and so its corresponding maximal operator admits $(H^p, L^{p,\infty})$-estimate for the critical index $\delta(p) = n(1/p - 1/2) - 1/2$ and $0 < p < 1$ as in that of Stein, Taibleson, and Weiss [10]. Thus we concentrate upon obtaining the decay estimate for the kernels $\mathcal{H}^\delta_{\varrho k\ell}$.

Lemma 2.1.. *For fixed $k \in \mathbb{N}$ and for $\ell = 1, 2, \cdots, N_k$, let $T_{\zeta_{k\ell}}(\Sigma_\varrho)$ be the tangent space of Σ_ϱ at $\zeta_{k\ell} \in \Sigma_\varrho$, $\{e^j_{k\ell}\}_{j=1}^{n-1}$ be an orthonormal basis of $T_{\zeta_{k\ell}}(\Sigma_\varrho)$, and $e^0_{k\ell}$ be the outer unit normal vector to Σ_ϱ at $\zeta_{k\ell} \in \Sigma_\varrho$. Then we have the following estimate*

$$\left|\mathcal{H}^\delta_{\varrho k\ell}(x)\right| \le \frac{C_N\, 2^{-k(\delta+1+(n-1)/2)}}{(1 + 2^{-k}|\langle x, e^0_{k\ell}\rangle|)^N \prod_{j=1}^{n-1}(1 + 2^{-k/2}|\langle x, e^j_{k\ell}\rangle|)^N}$$

for any $N \in \mathbb{N}$.

Proof. We need the following simple observation:

Let $\varrho \in C^N(\mathbb{R}^n)$ and $F \in C^N(\mathbb{R}_+)$. For $e \in S^{n-1}$, let $\mathcal{D}_e f$ be the directional derivative $\langle e, \nabla f\rangle$. Then one can have the formula (see [8])

$$\mathcal{D}^N_e (F \circ \varrho) = \sum_{\nu=1}^{N} F^{(\nu)} \circ \varrho \sum_{\beta \in \mathcal{Y}^N_\nu} \sum_{m=1}^{\nu} c_{N,\beta_m} \mathcal{D}^\beta_e \varrho \tag{2.1}$$

where $\mathcal{Y}^N_\nu = \{\beta |\ \sum_{m=1}^{\nu} \beta_m = N$, at least $\nu - \frac{N}{2}$ of the numbers β_m are equal to $1\}$, $\beta = (\beta_1, \cdots, \beta_\nu)$ is a multiindex, and c_{N,β_m}'s are some constants. For $k \in \mathbb{N}$, let $F_k(t) = \phi(2^{k+1}(1-t))(1-t)^\delta_+$. Then it follows from simple computation that

$$F^{(\nu)}_k(t) = (-1)^\nu \sum_{i=0}^{\nu} C(\nu, i) C(\delta, \nu - i) 2^{i(k+1)} \phi^{(i)}(2^{k+1}(1-t))(1-t)^{\delta-\nu+i} \tag{2.2}$$

where $C(\nu, i) = \nu(\nu-1)(\nu-2)\cdots(\nu-i+1)$ for positive integers ν, i, and $C(\nu, 0) = 1$. For fixed k, ℓ, we have the estimate

$$\left\|\mathcal{D}^N_{e^0_{k\ell}}[\Phi^\delta_k \Pi_{k\ell}]\right\|_{L^1} \le c\, 2^{-k(\frac{n+1}{2})} 2^{-k\delta} 2^{kN} \tag{2.3}$$

for any $N \in \mathbb{N}$. Since we have the better estimate $|\mathcal{D}_{e^j_{k\ell}}\varrho| \le c\,2^{-k/2}$ on the support of $\mathcal{F}[\mathcal{H}^\delta_{\varrho k\ell}]$ for fixed j,k,ℓ, it follows from (2.1) and Taylor's theorem that

$$\left\| \mathcal{D}^N_{e^j_{k\ell}}[\Phi^\delta_k \Pi_{k\ell}] \right\|_{L^1} \le c\,2^{-k(\frac{n+1}{2})} 2^{-k\delta} 2^{kN/2} \tag{2.4}$$

for any $N \in \mathbb{N}$. Using the integration by parts, it follows from (2.3) and (2.4) that

$$\left|\mathcal{H}^\delta_{\varrho k\ell}(x)\right| \le \frac{C_N\, 2^{-(\delta+1+(n-1)/2)k}}{(1+2^{-k}|\langle x, e^0_{k\ell}\rangle|)^N \prod_{j=1}^{n-1}(1+2^{-k/2}|\langle x, e^j_{k\ell}\rangle|)^N} \tag{2.5}$$

for any $N \in \mathbb{N}$. $\square$

We now introduce the real Hardy space $H^p(\mathbb{R}^n)$ defined in terms of atomic decompositions along the pattern of Stein [9]. For $0 < p \le 1$, a function $\mathfrak{a} \in L^\infty(\mathbb{R}^n)$ is called a (p,μ)-atom centered at $x_0 \in \mathbb{R}^n$ if it satisfies

(i) there is a ball $B(x_0; s)$ with supp $(\mathfrak{a}) \subset B(x_0; s)$,

(ii) $\|\mathfrak{a}\|_{L^\infty} \le |B(x_0;s)|^{-1/p}$, and

(iii) $\displaystyle\int_{\mathbb{R}^n} \mathfrak{a}(x) x^\alpha\, dx = 0$ for $|\alpha| \le \mu$,

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ is an n-tuple of nonnegative integers and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. If $f = \sum_{k=1}^\infty c_k \mathfrak{a}_k$ where the $\mathfrak{a}_k$'s are (p,μ)-atoms and $\{c_k\} \in \ell^p$, then $f \in H^p(\mathbb{R}^n)$ and $\|f\|^p_{H^p} \lesssim \sum_k |c_k|^p$ and the converse inequality also holds. Here we note that if $\delta > n(1/p - 1/2) - 1/2$ then $\mu = n(1/p' - 1)$ is enough for our oncoming estimates where $p' < p$ is a positive number satisfying $\delta = n(1/p' - 1/2) - 1/2$.

For $f \in \mathfrak{S}(\mathbb{R}^n)$, $\delta > 0$, $k \in \mathbb{N}$, and $\ell = 1, 2, \cdots, N_k$, let

$$\mathfrak{M}^\delta_{\varrho k\ell} f(x) = \sup_{t>0} \left| \mathcal{H}^{\delta,t}_{\varrho k\ell} * f(x) \right|$$

where $\mathcal{H}^{\delta,t}_{\varrho k\ell}(x) = t^n\, \mathcal{H}^\delta_{\varrho k\ell}(A^*_t x)$, and let $\mathfrak{M}^\delta_{\varrho k} f(x) = \sum_{\ell=1}^{N_k} \mathfrak{M}^\delta_{\varrho k\ell} f(x)$.

Lemma 2.2. *If $\delta > n(1/p - 1/2) - 1/2$ for $0 < p < 1$, let a positive number $p' < p$ be chosen so that $\delta = n(1/p' - 1/2) - 1/2$. For fixed $k \in \mathbb{N}$ and for $\ell = 1, 2, \cdots, N_k$, let $T_{\zeta_{k\ell}}(\Sigma_\varrho)$ be the tangent space of Σ_ϱ at $\zeta_{k\ell} \in \Sigma_\varrho$, $\{e^j_{k\ell}\}_{j=1}^{n-1}$ be an orthonormal basis of $T_{\zeta_{k\ell}}(\Sigma_\varrho)$, and $e^0_{k\ell}$ be the outer unit normal vector to Σ_ϱ at $\zeta_{k\ell} \in \Sigma_\varrho$. Then we have the following estimate*

$$\left|\mathcal{H}^\delta_{\varrho k\ell}(x)\right| + \left|\nabla \mathcal{H}^\delta_{\varrho k\ell}(x)\right| \le \frac{C_p\, 2^{-k(\frac{n-1}{2p'})}}{\prod_{j=0}^{n-1}(1+|\langle x, e^j_{k\ell}\rangle|)^{1/p'}} \fallingdotseq C_p\, 2^{-k(\frac{n-1}{2p'})} P_{k\ell}(x).$$

Proof. This can easily be obtained by choosing $\delta = n(1/p' - 1/2) - 1/2$ and $N = 1/p'$ in Lemma 2.1. We also observe that $\nabla \mathcal{H}^\delta_{\varrho k\ell} = \varphi * \mathcal{H}^\delta_{\varrho k\ell}$ for some $\varphi \in \mathfrak{S}(\mathbb{R}^n)$. $\square$

Lemma 2.3. *If $\delta > n(1/p - 1/2) - 1/2$ for $0 < p < 1$, let a positive number $p' < p$ be chosen so that $\delta = n(1/p' - 1/2) - 1/2$. Suppose that $\mathfrak{a}$ is a $(p, n(1/p' - 1))$-atom on $\mathbb{R}^n$ which is supported in the ball $B(x_0; s)$ with center $x_0 \in \mathbb{R}^n$ and radius $s > 0$. Then there is a constant $C = C(n,p) > 0$ such that*

(a) $\left|\mathfrak{M}^\delta_{\varrho k\ell}\mathfrak{a}(x)\right| \le C\, s^{-n/p}\, 2^{-k(\frac{n-1}{2p'})} P_{k\ell}\left(\dfrac{x - x_0}{s}\right)$ *for any* $x \in B(x_0; 2s)^c$,

(b) $\left\|(\mathfrak{M}^{\delta}_{\varrho k\ell}\mathfrak{a})\chi_{B(x_0;2s)^c}\right\|_{L^p} \le C\, 2^{-k(\frac{n-1}{2p'})}$,

where $P_{k\ell}(x)$ *is the function given in Lemma 2.2.*

Proof. (a) We first assume that $\mathfrak{a}$ is a $(p, n(1/p'-1))$-atom which is supported in the unit ball $B(0;1)$ centered at the origin and let $N \in \mathbb{N}$ be an integer satisfying $N < n(1/p'-1) \le N+1$, i.e. $n/(n+N+1) \le p' < n/(n+N)$. If $x \in B(0;2)^c$ and $t > 1$, then it easily follows from Lemma 2.2 that

$$\left|\mathcal{H}^{\delta,t}_{\varrho k\ell} * \mathfrak{a}(x)\right| \le C\, t^{n(1-1/p')}\, 2^{-k(\frac{n-1}{2p'})} P_{k\ell}(x).$$

Since $n(1-1/p') < 0$, we have that

$$\sup_{t>1}\left|\mathcal{H}^{\delta,t}_{\varrho k\ell} * \mathfrak{a}(x)\right| \le C\, 2^{-k(\frac{n-1}{2p'})} P_{k\ell}(x). \tag{2.6}$$

If $x \in B(0;2)^c$ and $0 < t \le 1$, let $\mathcal{Q}_{t,x}(y)$ be the N-th order Taylor polynomial of the function $y \mapsto \mathcal{H}^{\delta(p)}_{\varrho k\ell}(A^*_t(x-y))$ expanded near the origin. Using the moment conditions on the atom $\mathfrak{a}$ and Taylor's theorem, we obtain the estimate

$$\begin{aligned}
\left|\mathfrak{M}^{\delta,t}_{\varrho k\ell} * \mathfrak{a}(x)\right| &= t^n \left|\int_{\mathbb{R}^n} [\mathcal{H}^{\delta}_{\varrho k\ell}(A^*_t(x-y)) - \mathcal{Q}_{t,x}(y)]\mathfrak{a}(y)\, dy\right| \\
&\lesssim t^{n+(N+1)} \int_0^1 \int_{B(0;1)} \left|\nabla^{N+1}\mathcal{H}^{\delta}_{\varrho k\ell}(A^*_t(x-\tau y))\right|\, dy\, d\tau \\
&\lesssim t^{n+(N+1)-n/p'}\, 2^{-k(\frac{n-1}{2p'})} P_{k\ell}(x)
\end{aligned}$$

because $n+(N+1)-n/p' \ge 0$. Thus we have that

$$\sup_{0<t\le 1}\left|\mathcal{H}^{\delta,t}_{\varrho k\ell} * \mathfrak{a}(x)\right| \lesssim 2^{-k(\frac{n-1}{2p'})} P_{k\ell}(x). \tag{2.7}$$

By (2.6) and (2.7) we have that $\mathfrak{M}^{\delta}_{\varrho k\ell}\mathfrak{a}(x) \lesssim 2^{-k(\frac{n-1}{2p'})} P_{k\ell}(x)$.

Finally, let $\mathfrak{a}$ be a $(p, n(1/p'-1))$-atom which is supported in that ball $B(x_0;s)$. Without loss of generality, we assume that $x_0 = 0$. Let $\mathfrak{b}(x) = s^{n/p}\,\mathfrak{a}(A_s\, x)$. Then $\mathfrak{b}$ is clearly a $(p, n(1/p'-1))$-atom supported in the unit ball $B(0;1)$. We also observe that

$$\begin{aligned}
\mathcal{H}^{\delta,1/t}_{\varrho k\ell} * \mathfrak{a}(x) &= \int_{\mathbb{R}^n} \mathcal{H}^{\delta}_{\varrho k\ell}(A_{1/t}\, x - y)\, \mathfrak{a}(A_t\, y)\, dy \\
&= s^{-n/p} \int_{\mathbb{R}^n} \mathcal{H}^{\delta}_{\varrho k\ell}(A_{s/t}A_{1/s}\, x - y)\, \mathfrak{b}(A_{t/s}\, y)\, dy \\
&= s^{-n/p}(t/s)^{-n} \int_{\mathbb{R}^n} \mathcal{H}^{\delta}_{\varrho k\ell}(A_{s/t}(A_{1/s}\, x - y))\, \mathfrak{b}(y)\, dy \\
&= s^{-n/p}\mathcal{H}^{\delta,s/t}_{\varrho k\ell} * \mathfrak{b}(A_{1/s}\, x).
\end{aligned} \tag{2.8}$$

Therefore, combining this with the above estimate, we complete the part (a).

(b) We observe that there is a constant $C = C(n,p) > 0$ such that for any $x_0 \in \mathbb{R}^n$ and for any $k \in \mathbb{N}$, $\ell = 1, 2, \cdots, N_k$,

$$\|P_{k\ell}(\cdot - x_0)\|_{L^p} \le C. \tag{2.9}$$

Then it easily follows from the change of variable and (2.9) that

$$\left\|(\mathfrak{M}^{\delta}_{\varrho k\ell}\mathfrak{a})\chi_{B(x_0;2s)^c}\right\|_{L^p} \le C\,2^{-k(\frac{n-1}{2p'})}\|P_{k\ell}(\cdot - x_0/s)\|_{L^p} \le C\,2^{-k(\frac{n-1}{2p'})}. \quad \square$$

Proof of Theorem 1.2. First of all, we prove that if $\delta > n(1/p-1/2)-1/2$ for $0<p<1$ then $\mathfrak{M}^{\delta}_{\varrho}\mathfrak{a} \in L^p(\mathbb{R}^n)$ for any $(p, n(1/p'-1))$-atom on $\mathbb{R}^n$ where $p'<p$ is a positive number satisfying $\delta = n(1/p'-1/2)-1/2$, and moreover there is a constant $C>0$ independent of such atoms such that $\|\mathfrak{M}^{\delta}_{\varrho}\mathfrak{a}\|_{L^p} \le C$. For $t>0$ and $\delta>0$, let $\mathcal{H}^{\delta}_{\varrho,t}(x) = \mathcal{F}^{-1}[(1-\varrho/t)^{\delta}_{+}](x)$ and let $\mathcal{H}^{\delta}_{\varrho,1}(x) = \mathcal{H}^{\delta}_{\varrho}(x)$. Let $\mathfrak{a}$ be a $(p, n(1/p'-1))$-atom supported in the ball $B(x_0;s)$ with center $x_0\in\mathbb{R}^n$ and radius $s>0$. Then we see that $\mathfrak{R}^{\delta}_{\varrho,t}\mathfrak{a}(x) = \mathcal{H}^{\delta}_{\varrho,t} * \mathfrak{a}(x)$. Since $\mathcal{H}^{\delta}_{\varrho}\in L^1(\mathbb{R}^n)$ by Lemma 2.2, if $x\in B(0;2s)$ is given then we have that

$$\left|\mathfrak{R}^{\delta}_{\varrho,t}\mathfrak{a}(x)\right| \le \left\|\mathcal{H}^{\delta}_{\varrho,t}\right\|_{L^1}\,\|\mathfrak{a}\|_{L^\infty} \le \left\|\mathcal{H}^{\delta}_{\varrho}\right\|_{L^1}\,|B(x_0;s)|^{-1/p},$$

and so

$$\mathfrak{M}^{\delta}_{\varrho}\mathfrak{a}(x) \lesssim |B(x_0;s)|^{-1/p}.$$

Since $0<p<1$, it easily follows from (b) of Lemma 2.3 that

$$\begin{aligned} \|\mathfrak{M}^{\delta}_{\varrho}\mathfrak{a}\|^p_{L^p} &= \|(\mathfrak{M}^{\delta}_{\varrho}\mathfrak{a})\chi_{B(x_0;2s)}\|^p_{L^p} + \|(\mathfrak{M}^{\delta}_{\varrho}\mathfrak{a})\chi_{B(x_0;2s)^c}\|^p_{L^p} \\ &\le 2^n + \sum_{k=1}^{\infty}\sum_{\ell=1}^{N_k}\|(\mathfrak{M}^{\delta}_{\varrho k\ell}\mathfrak{a})\chi_{B(x_0;2s)^c}\|^p_{L^p} \\ &\lesssim 2^n + C\sum_{k=1}^{\infty}2^{-k(\frac{p}{p'}-1)(\frac{n-1}{2})} \le C. \end{aligned} \tag{2.10}$$

Finally, if $f=\sum_{j=1}^{\infty}c_j\mathfrak{a}_j$ where the $\mathfrak{a}_j$'s are $(p,n(1/p'-1))$-atoms and $\{c_j\}\in\ell^p$, then by (2.10) we have the estimate

$$\|\mathfrak{M}^{\delta}_{\varrho}f\|^p_{L^p} \le \sum_j |c_j|^p\,\|\mathfrak{M}^{\delta}_{\varrho}\mathfrak{a}_j\|^p_{L^p} \lesssim \sum_j |c_j|^p.$$

Hence this completes the proof. $\square$

3. $(H^p, L^{p,\infty})$-estimate for the case that Σ_ϱ is a smooth convex hypersurface of finite type.

In this section we shall focus upon obtaining $(H^p,L^{p,\infty})$-mapping properties of the maximal operator $\mathfrak{M}^{\delta(p)}_{\varrho}$, $p<1$, under the condition that Σ_ϱ is a smooth convex hypersurface of finite type.

Let Σ be a smooth convex hypersurface of $\mathbb{R}^n$ and let $d\sigma$ be the induced surface area measure on Σ. Let $\mathcal{E}(\Sigma)$ be the set of points of Σ at which the Gaussian curvature κ vanishes, and let $\mathcal{N}(\Sigma)=\{n(\xi)\,|\,\xi\in\mathcal{E}(\Sigma)\}$ where $n(\xi)$ denotes the outer unit normal to Σ at $\xi\in\Sigma$. For $x\in\mathbb{R}^n$, denote by $d(x/|x|,\mathcal{N}(\Sigma))$ the geodesic distance on S^{n-1} between $x/|x|$ and $\mathcal{N}(\Sigma)$, and by $\mathcal{B}(\xi(x),s)$ the spherical cap near $\xi(x)\in\Sigma$ cut off from Σ by a plane parallel to $T_{\xi(x)}(\Sigma)$ (the affine tangent plane to Σ at $\xi(x)$) at distance $s>0$ from it; that is,

$$\mathcal{B}(\xi(x),s) = \{\xi\in\Sigma\,|\,d(\xi,T_{\xi(x)}(\Sigma))<s\},$$

where $\xi(x)$ is the point of Σ whose outer unit normal is in the direction x. These spherical caps play an important role in furnishing the decay of the Fourier transform of the measure $d\sigma$. It is well known [7,9] that the function

$$\Omega(\theta) \doteqdot \sup_{r>0}\sigma[\mathcal{B}(\xi(r\theta),1/r)](1+r)^{\frac{n-1}{2}} \tag{3.1}$$

is bounded on S^{n-1} provided that Σ has nonvanishing Gaussian curvature.

Definition 3.1. *Σ be a smooth convex hypersurface of $\mathbb{R}^n$. Then we say that Σ satisfies a spherically integrable condition of order < 1 if $\Omega \in L^p(S^{n-1})$ for any $p < 1$.*

Remark. (i) B. Randol [7] proved that if Σ is a real analytic convex hypersurface of $\mathbb{R}^n$ then $\Omega \in L^p(S^{n-1})$ for some $p > 2$. Thus any real analytic convex hypersurface satifies a spherically integrable condition of order < 1.

(ii) Let Σ be a smooth convex hypersurface of finite type $k \geq 2$ and suppose that $\mathcal{N}(\Sigma)$ is a m-dimensional submanifold of $\mathbb{R}^n$ which is on S^{n-1}, where $m < [k(n-1)]/[2(k-1)]$. Then we see (refer to [4]) that Σ satisfies a spherically integrable condition of order < 1. Moreover, it is not hard to see that Σ satisfies a spherically integrable condition even for $m \leq n-2$. We mention for reader that it can be shown by Lemma 2.8 [4] and the fact Σ is of finite type $P(k)$; i.e. there is some constant $C = C(\Sigma) > 0$ such that for any $\theta \in S^{n-1}$,

$$\Omega(\theta) \leq \frac{C}{d(\theta, \mathcal{N}(\Sigma))^{\frac{k-2}{2(k-1)}(n-1)}}.$$

Since Σ is smooth and of finite type, it is absolutely impossible that $\mathcal{N}(\Sigma)$ is a $(n-1)$-dimensional submanifold of $\mathbb{R}^n$ which is on S^{n-1}.

(iii) More generally, it was shown by I. Svensson [13] that if Σ is a smooth convex hypersurface of finite type $k \geq 2$ then $\Omega \in L^p(S^{n-1})$ for some $p > 2$.

Thus, by the above remark (iii), it is natural for us to obtain the following lemma.

Lemma 3.2. *Any smooth convex hypersurface of finite type always satisfies a spherically integrable condition of order < 1.*

Sharp decay estimates for the Fourier transform of surface measure on a smooth convex hypersurface Σ of finite type $k \geq 2$ has been obtained by Bruna, Nagel, and Wainger [1]; precisely speaking, $|\mathcal{F}[d\sigma](x)|$ is equivalent to $\sigma[\mathcal{B}(\xi(x), 1/|x|)]$. They define a family of anisotropic balls on Σ by letting

$$\mathcal{B}(\xi_0, s) = \{\xi \in \Sigma |\, d(\xi, T_{\xi_0}(\Sigma)) < s\}$$

where $\xi_0 \in \Sigma$. We now recall some properties of the anisotropic balls $\mathcal{B}(\xi_0, s)$ associated with Σ. The proof of the doubling property in [1] makes it possible to obtain the following stronger estimate for the surface measure of these balls;

$$\sigma[\mathcal{B}(\xi_0, \gamma s)] \lesssim \begin{cases} \gamma^{\frac{n-1}{2}} \sigma[\mathcal{B}(\xi_0, s)], & \gamma \geq 1, \\ \gamma^{\frac{n-1}{k}} \sigma[\mathcal{B}(\xi_0, s)], & \gamma < 1. \end{cases} \tag{3.2}$$

It also follows from the triangle inequality and the doubling property [1] that there is a positive constant $C > 0$ independent of $s > 0$ such that

$$\frac{1}{C}\, \sigma[\mathcal{B}(\xi_0, s)] \leq \sigma[\mathcal{B}(\xi, s)] \leq C\, \sigma[\mathcal{B}(\xi_0, s)] \quad \text{for any } \xi \in \mathcal{B}(\xi_0, s). \tag{3.3}$$

Next we recall a useful lemma [10] due to E. M. Stein, M. H. Taibleson, and G. Weiss on summing up weak type functions.

Lemma 3.3. *Let $0 < p < 1$. Suppose that $\{\mathfrak{h}_k\}$ is a sequence of measurable functions such that for all $k \in \mathbb{N}$,*

$$\|\mathfrak{h}_k\|_{L^{p,\infty}} \leq 1.$$

If $\{c_k\} \in \ell^p$*, then we have the following estimate*

$$\left\|\sum_{k=1}^{\infty} c_k \mathfrak{h}_k\right\|_{L^{p,\infty}} \leq \left(\frac{2-p}{1-p}\right)^{1/p} \|\{c_k\}\|_{\ell^p}.$$

We now state an elementary lemma without proof which will be useful to measure the distance from a point of $\mathcal{B}(\xi_0, s)$ to the affine tangent plane to Σ at $\xi_0 \in \Sigma$ in higher dimensions.

Lemma 3.4. *Let* Σ *be a smooth simple closed convex curve in* $\mathbb{R}^2$ *whose graph near* $(0,0)$ *is given as* $(t, g(t))$ *where* $g(t) = b\,t^m + c$ *is a convex function defined on* $[-d, d]$ *for some sufficiently small constant* $b, c, d > 0$ *and an integer* $m \geq 2$*. For* $|t| \leq d$*, we denote by* $\Theta(t)$ *the angle between* $n(0, g(0))$ *and* $n(t, g(t))$*. For some small angle* $\Theta_0 > 0$ *with* $\Theta_0 \leq \max\{\Theta(-d), \Theta(d)\}$*, let* t_0 *be chosen so that* $\Theta(t_0) = \Theta_0$ *and* $|t_0| \leq d$*. Then we have the following estimate*

$$|g(t_0) - c| \sim |b|^{-\frac{1}{m-1}} m^{-\frac{m}{m-1}} \Theta_0^{\frac{m}{m-1}}.$$

Lemma 3.5. *Let* Σ *be a smooth convex hypersurface of* $\mathbb{R}^n$ *which is of finite type* $k \geq 2$*. Then there is a constant* $C = C(\Sigma) > 0$ *such that for any* $y \in B(0; s)$ *and* $x \in B(0; 2s)^c$, $0 < s \leq 1$,

$$\xi(x-y) \in \mathcal{B}(\xi(x), C/|x|)$$

where $\xi(x)$ *is the point of* Σ *whose outer unit normal is in the direction* x*.*

Proof. We observe that the following inequality always holds for any $x, y \in \mathbb{R}^n$ with $|x| > 2|y|$;

$$\left|\frac{x-y}{|x-y|} - \frac{x}{|x|}\right| \leq 2\,\frac{|y|}{|x|}. \tag{3.4}$$

Near $\xi(x/|x|) \in \Sigma$, the hypersurface Σ can be given as the graph of a smooth convex function defined on $D \risingdotseq T_{\xi(x/|x|)}(\Sigma) \cap B(\xi(x/|x|); 1/2)$; to be precise, let Ψ be a smooth convex function defined on D such that $(\xi_0', \Psi(\xi_0')) = \xi(x/|x|)$, and for $|t| < 1/2$ and $\eta \in T^{n-1} \risingdotseq [T_{\xi(x/|x|)}(\Sigma) - \xi(x/|x|)] \cap S^{n-1}$,

$$\Psi(\xi_0' + t\eta) = \sum_{i=0}^{k} \frac{1}{i!}\,\mathcal{D}_\eta^i \Psi(\xi_0')\,t^i + \mathcal{O}(t^{k+1}). \tag{3.5}$$

Using (3.5), we now estimate the distance from $\xi = (\xi', \Psi(\xi')) \in \Sigma$ to the tangent space $T_{\xi(x/|x|)}(\Sigma)$ as follows; since Σ is of finite type $k \geq 2$, for each $\eta \in T^{n-1}$ there is an integer m with $2 \leq m \leq k$ such that for $-1/2 < t < 1/2$

$$\Psi(\xi_0' + t\eta) - \Psi(\xi_0') - \mathcal{D}_\eta \Psi(\xi_0')\,t = \frac{1}{m!}\,\mathcal{D}_\eta^m \Psi(\xi_0')\,t^m + \mathcal{O}(t^{m+1}).$$

Thus by (3.4) and Lemma 3.4 we have that

$$\begin{aligned}
\left\langle \xi\left(\frac{x}{|x|}\right) - \xi\left(\frac{x-y}{|x-y|}\right), \frac{x}{|x|} \right\rangle &= \Psi(\xi_0' + t_1\eta) - \Psi(\xi_0') - \mathcal{D}_\eta \Psi(\xi_0')\,t_1 \\
&\lesssim \left[\frac{m^m}{m!}\,|\mathcal{D}_\eta^m \Psi(\xi_0')|\right]^{-\frac{1}{m-1}} \left|\frac{x-y}{|x-y|} - \frac{x}{|x|}\right|^{\frac{m}{m-1}} \\
&\leq M_0 \left|\frac{x-y}{|x-y|} - \frac{x}{|x|}\right| \leq \frac{2M_0}{|x|}
\end{aligned}$$

where t_1, $|t_1| < 1/2$, is some number so that $(\xi_0' + t_1\,\eta, \Psi(\xi_0' + t_1\,\eta)) = \xi\left(\frac{x-y}{|x-y|}\right)$ and $M_0 = \sup_{2\le m\le k}\sup_{\eta\in T^{n-1}}\left[\frac{m^m}{m!}\,|\mathcal{D}_\eta^m\Psi(\xi_0')|\right]^{-\frac{1}{m-1}}$. Hence we complete the proof. $\square$

Lemma 3.6. *Let Σ be a smooth convex hypersurface of $\mathbb{R}^n$ which is of finite type $k \geq 2$. Then there is a constant $C = C(\Sigma) > 0$ such that for any $y \in B(0;s)$ and $x \in B(0;2s)^c$, $0 < s \leq 1$,*

$$\Omega\left(\frac{x-y}{|x-y|}\right) \leq C\,\Omega\left(\frac{x}{|x|}\right)$$

where Ω is the radial function defined as in (3.1).

Proof. It easily follows from (3.2), (3.3), the definition of Ω, and Lemma 3.5 that for any $y \in B(0;s)$ and $x \in B(0;2s)^c$, $0 < s \leq 1$,

$$\begin{aligned}\Omega\left(\frac{x-y}{|x-y|}\right) &= \sup_{r>0}\sigma[\mathcal{B}(\xi(x-y),1/r)]\,(1+r)^{\frac{n-1}{2}}\\ &\lesssim \sup_{r>0}\sigma[\mathcal{B}(\xi(x),1/r)]\,(1+r)^{\frac{n-1}{2}} = \Omega\left(\frac{x}{|x|}\right). \qquad \square\end{aligned}$$

Proof of Theorem 1.1. Fix $0 < p < 1$. Let $\mathfrak{a}$ be a $(p, n(1/p-1))$-atom supported in the ball $B(x_0;s)$ with center $x_0 \in \mathbb{R}^n$ and radius $s > 0$. Then we see that $\mathfrak{R}^\delta_{\varrho,t}\mathfrak{a}(x) = \mathcal{H}^\delta_{\varrho,t} * \mathfrak{a}(x)$. Recalling the lemma [6] about asymptotics of quasiradial Bochner-Riesz kernel and the result of Bruna, Nagel, and Wainger [1], we get that

$$\left|\mathcal{H}_\varrho^{\delta(p)}(x)\right| \sim \left|\nabla\mathcal{H}_\varrho^{\delta(p)}(x)\right| \sim \frac{1}{(1+|x|)^{\frac{n}{p}-\frac{n-1}{2}}}\,\sigma[\mathcal{B}(\xi(x),1/|x|)] \tag{3.6}$$

where we consider Σ_ϱ as Σ given in the above. Since $\mathcal{H}_\varrho^{\delta(p)} \in L^1(\mathbb{R}^n)$ by (3.6) and Lemma 3.2, if $x \in B(0;2s)$ is given then we have that

$$\left|\mathfrak{R}^{\delta(p)}_{\varrho,t}\mathfrak{a}(x)\right| \leq \left\|\mathcal{H}^{\delta(p)}_{\varrho,t}\right\|_{L^1}\|\mathfrak{a}\|_{L^\infty} \leq \left\|\mathcal{H}^{\delta(p)}_{\varrho}\right\|_{L^1}|B(x_0;s)|^{-1/p},$$

and so

$$\mathfrak{M}_\varrho^{\delta(p)}\mathfrak{a}(x) \lesssim |B(x_0;s)|^{-1/p}.$$

Thus we have that for all $\lambda > 0$,

$$\left|\{x \in B(x_0;2s)|\,\mathfrak{M}_\varrho^{\delta(p)}\mathfrak{a}(x) > \lambda/2\}\right| \lesssim \lambda^{-p}. \tag{3.7}$$

Next we shall obtain the following inequality

$$\left|\{x \in B(x_0;2s)^c|\,\mathfrak{M}_\varrho^{\delta(p)}\mathfrak{a}(x) > \lambda/2\}\right| \lesssim \lambda^{-p},\ \lambda > 0. \tag{3.8}$$

As in the argument of (2.8), without loss of generality we can assume that a $(p, n(1/p-1))$-atom $\mathfrak{a}$ is supported in the unit ball $B(0;1)$ centered at the origin.

We now consider the case that $x \in B(0;2)^c$ and $t > 1$. Then it follows from (3.1), (3.2), (3.6), and Lemma 3.6 that

$$\begin{aligned}\left|\mathcal{H}^{\delta(p)}_{\varrho,t} * \mathfrak{a}(x)\right| &\lesssim t^n \int_{B(0;1)} |\mathcal{H}^{\delta(p)}_{\varrho}(A_t(x-y))|\,dy \\ &\lesssim \frac{t^{n-n/p}}{(1+|x|)^{\frac{n}{p}}} \int_{B(0;1)} \Omega\left(\frac{x-y}{|x-y|}\right)\,dy \\ &\lesssim \frac{t^{n-n/p}}{(1+|x|)^{\frac{n}{p}}}\,\Omega\left(\frac{x}{|x|}\right) \\ &\lesssim \frac{1}{(1+|x|)^{\frac{n}{p}}}\,\Omega\left(\frac{x}{|x|}\right)\end{aligned}$$

because $n(1-1/p) < 0$. So we have that

$$\sup_{t>1} \left|\mathcal{H}^{\delta(p)}_{\varrho,t} * \mathfrak{a}(x)\right| \lesssim \frac{1}{(1+|x|)^{\frac{n}{p}}}\,\Omega\left(\frac{x}{|x|}\right). \tag{3.9}$$

Let $N \in \mathbb{N}$ be an integer satisfying $N < n(1/p-1) \leq N+1$, i.e. $n/(n+N+1) \leq p < n/(n+N)$. If $x \in B(0;2)^c$ and $0 < t \leq 1$, let $\mathcal{Q}_{t,x}(y)$ be the N-th order Taylor polynomial of the function $y \mapsto \mathcal{H}^{\delta(p)}_{\varrho}(A^*_t(x-y))$ expanded near the origin, where $\mathcal{H}^{\delta(p)}_{\varrho}(x) = \mathcal{F}^{-1}[(1-\varrho)^{\delta(p)}_{+}](x)$. Then it follows from the moment condition on the atom $\mathfrak{a}$, Taylor's theorem, (3.1), (3.2), (3.6), and Lemma 3.6 that

$$\begin{aligned}\left|\mathcal{H}^{\delta(p)}_{\varrho,t} * \mathfrak{a}(x)\right| &= t^n \left|\int_{\mathbb{R}^n} [\mathcal{H}^{\delta(p)}_{\varrho}(A_t(x-y)) - \mathcal{Q}_{t,x}(y)]\mathfrak{a}(y)\,dy\right| \\ &\lesssim t^{n+(N+1)} \int_0^1 \int_{B(0;1)} |\nabla^{N+1}\mathcal{H}^{\delta(p)}_{\varrho}(A_t(x-\tau y))|\,dy\,d\tau \\ &\lesssim \frac{t^{n+(N+1)-n/p}}{(1+|x|)^{\frac{n}{p}}} \int_0^1 \int_{B(0;1)} \Omega\left(\frac{x-\tau y}{|x-\tau y|}\right)\,dy\,d\tau \\ &\lesssim \frac{t^{n+(N+1)-n/p}}{(1+|x|)^{\frac{n}{p}}}\,\Omega\left(\frac{x}{|x|}\right) \\ &\lesssim \frac{1}{(1+|x|)^{\frac{n}{p}}}\,\Omega\left(\frac{x}{|x|}\right)\end{aligned}$$

because $n+(N+1)-n/p \geq 0$. Thus we have that

$$\sup_{0<t\leq 1} \left|\mathcal{H}^{\delta(p)}_{\varrho,t} * \mathfrak{a}(x)\right| \lesssim \frac{1}{(1+|x|)^{\frac{n}{p}}}\,\Omega\left(\frac{x}{|x|}\right). \tag{3.10}$$

Thus by (3.9) and (3.10) we conclude that

$$\mathfrak{M}^{\delta(p)}_{\varrho}\mathfrak{a}(r\theta) \lesssim \frac{1}{(1+r)^{\frac{n}{p}}}\,\Omega(\theta).$$

Hence we have the following estimate

$$\int_{\{x\in B(0;2)^c \,|\, \mathfrak{M}^{\delta(p)}_{\varrho}\mathfrak{a}(x)>\lambda\}} dx \lesssim \int_{S^{n-1}} \int_{\{r>0 \,|\, 2<r<\lambda^{-p/n}\,\Omega(\theta)^{p/n}\}} r^{n-1}\,dr\,d\theta \lesssim \lambda^{-p}$$

because $\Omega \in L^p(S^{n-1})$ for any $p < 1$ by Lemma 3.2. Therefore, by (3.7), (3.8), and Lemma 3.3, we complete the proof. $\square$

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